The Fundamental Theorem of Calculus, Part I

Given a continuous function f on an interval I and a fixed point $a \in I$, define the function F on I by $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$ for all $x \in I$.

Example 1.

Find the derivative of the function defined by

$$F(x) = \int_{10}^{x} 3\left(t^2 + \sqrt{t}\right)dt$$

$f(t) = 3(t^2 + \sqrt{t})$

f is continuous on $[10, \infty)$. So, $F'(x) = 3(x^2 + \sqrt{x})$ for $x \geq 10$

Example 2.

Find the derivative of the function defined by

$$F(x) = \int_1^{2x^3} \left(1 + \sin^2 t\right) dt$$

Let $G(u) = \int_1^u (1 + \sin^2 t)\,dt$.

Now $G'(u) = 1 + \sin^2 u$ and $F(x) = G(2x^3)$.

So $F'(x) = G'(2x^3) \cdot 6x^2$ (chain rule)

$F'(x) = \left[1 + \sin^2(2x^3)\right] \cdot 6x^2$

The Fundamental Theorem Part II

If f is a continuous function on the interval [a,b] and if F is any antiderivative of f *on* $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Evaluate $\int_1^2 \frac{4x^5 - 7x^2}{x^3} dx$ $\quad f(x) = \frac{4x^5 - 7x^2}{x^3} = 4x^2 - \frac{7}{x}$

An antiderivative is $F(x) = \frac{4x^3}{3} - 7\ln x$.

$$\int_1^2 \frac{4x^5 - 7x^2}{x^3} dx = \left(\frac{4x^3}{3} - 7\ln x\right)\Big|_1^2$$

$$= \left(\frac{32}{3} - 7\ln 2\right) - \left(\frac{4}{3} - 0\right) = \frac{28}{3} - 7\ln 2$$

Evaluate $\int_0^{\frac{\pi}{4}} \left(3\sin\theta + 2\sec^2\theta\right) d\theta$.

An antiderivative of $3\sin\theta + 2\sec^2\theta$

is $-3\cos\theta + 2\tan\theta$.

$$\left(-3\cos\theta + 2\tan\theta\right)\Big|_0^{\frac{\pi}{4}} = \left(-\frac{3\sqrt{2}}{2} + 2\right) - \left(-3 + 0\right)$$

$$= -\frac{3\sqrt{2}}{2} + 5$$